Numbers and Infinity

Adrian Hall, March 2013 for U3A Cheltenham Science and Technology Group

1. Joke

An Engineer, a Physicist and a Mathematician are travelling north by train from England. Soon after entering Scotland (which none of them have previously visited) they see a black sheep in a field. The Engineer says “So - the sheep in Scotland are black”. The Physicist says “No – all we can say is that some of the sheep in Scotland are black”. The Mathematician says “There exists at least one field in Scotland in which there is at least one sheep at least half of which is black . . .”

Although this unfairly characterises Engineers and Physicists (for whom I have the highest respect) there is an element of truth in the caricature of Mathematicians who are notorious pedants in their search for rigour.

2. Natural Numbers and Systems of Axioms

Many mathematicians like to start from “the basics” and build up structures from a minimal number of assumptions. In one such scheme the most basic building blocks are the “Natural Numbers” – these are the numbers used for counting - 1, 2, 3, . . . (where the dots indicate that we can continue to add as many as we wish. In a simple sense, then, there is an unending sequence of natural numbers - an **infinite** number of them. Clearly, if anyone claims to have found the “largest natural number” we can always add one to it to get a larger one.

There is a set of generally agreed “**AXIOMS**” for the natural numbers – that is a set of assumptions that cannot be proved, but are so basic and “self-evident” that we are happy to use them as a basis for proving everything else we need. There are nine of them and they are called the “Peano Axioms” after the Italian mathematician who codified them. We do not have time to delve into them in any detail, but here are three (for flavour - liberties have been taken):

- 1 is a natural number;
- Every natural number has a successor;
- The number 1 has no predecessor . . .

From the Peano Axioms one can derive all of the properties of arithmetic as we know it, including such gems as 2 x 3 = 6 (where 2 is the successor to 1, 3 is the successor to 2 and 6 is the successor to the successor to the successor to the successor to the successor to 1).

Mathematicians lie awake at night worrying about axiom systems: Are they:

(a) CONSISTENT? (in other words are there any internal contradictions?);

(b) INDEPENDENT? (we would hate to have an axiom which is redundant because it could be derived from the others;

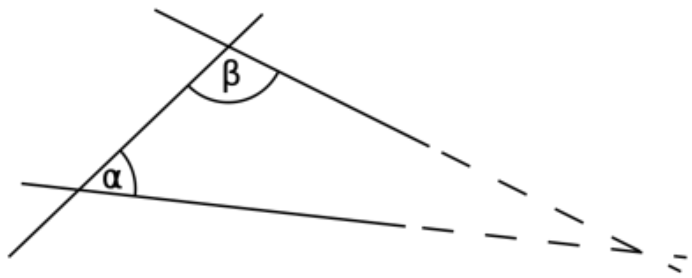
(c) COMPLETE? (in the sense that we can derive from them, using only rigorous logic, all of the properties and theorems that we would expect?).

Probably the most famous system of axioms is that proposed by the Ancient Greek Mathematician Euclid (in his book “Elements”) to underpin plane Geometry. They are in many ways unsatisfactory from the modern perspective, but are a triumph for their day.

He starts with five “Postulates” which assert that the following geometrical constructions are possible:

1. "To draw a straight line from any point to any point."
2. "To produce (extend) a finite straight line continuously in a straight line."
3. "To describe a circle with any centre and distance (radius)."
4. "That all right angles are equal to one another."
5. *The parallel postulate:* "That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles."

   (Roughly speaking, two lines which are not parallel will eventually intersect).



The *Elements* also include the following five "common notions":

1. Things that are equal to the same thing are also equal to one another (transitive property of equality).
2. If equals are added to equals, then the wholes are equal.
3. If equals are subtracted from equals, then the remainders are equal.
4. Things that coincide with one another equal one another (Reflexive Property).
5. The whole is greater than the part.

Using only these “axioms” the whole of plane geometry can be constructed – including such theorems as “The angles at the base of an isosceles triangle are equal”.

Interestingly, the Parallel Postulate worried Euclid greatly. Much of plane geometry can be constructed without it – but he included it to be able to prove certain "known" things about the world of plane geometry that he could not prove without it. It is now known that this postulate can be replaced with others to produce other self-consistent geometries.

Note particularly "Common Notion" number 5 - "The whole is greater than the part" which we shall challenge when we come to consider infinite sets.

3. Sets and Set Theory

Now the word "Set" is out of the bag we had better look at a few bits of elementary set theory.

A "**set**" is a collection of distinct objects called its "**elements**".

A set can be defined by listing its elements, for example

{ 3, 7, 2, 5 }

or

{ Bed, Table, Chair }

where the braces are used only to enclose the elements and commas are traditionally used to separate the elements. If the list is finite, but large, it can be helpful to use the "ellipsis" (a set of three dots) to indicate that some elements have been left out. For example the set of all the lower case traditional letters in English could be abbreviated to { a, b, c, . . . , z }. This same shorthand can be also helpful in the case of infinite sets, for example:

{ 2, 4, 6, 8, . . . } is the set of all even positive whole numbers

and

{ 2, 4, 8, 16, . . . } is the set of all positive integral powers of two.

There is a conventional notation to specify a set defined by a "rule" - for example we can write:

{ p: p is a prime number }.

Read this as "the set of all elements p, such that p is a prime number". The use of "p" in this sentence is as a "dummy variable" we could equally well have used "x" or any other such symbol. Note that this set is infinite (as proved in my last lecture to this group, some five years ago!)

We shall generally use capital letters to refer to sets, so we can write, for example,

P = { p: p is a prime number }.

I shall write N(A) for the number of elements in the finite set A. For example if A ={ a, e, i, o, u } then N(A) = 5.

We use the symbol "∈" as a shorthand for "belongs to" or "is an element of" or "is a member of".

Examples are:

a ∈ { h, a, p, l, e, s },

17 ∈ { p: p is a prime number }, (17 belongs to the set of prime numbers).

It is sometimes useful to be able to say that a particular element is *not* a member of a set, and the symbol ∉ is used for "is not an element of". For example: 6 ∉ { p: p is a prime number }.

Note that the order of the elements in a set is unimportant – so the sets

{ b, a, d }

and

{ d, a, b}

are the same set. In fact we say two sets are "**equal"** if they have precisely the same elements and use the "=" sign to denote this. The "≠" (not equal) sign can also be useful. Hence, for example:

{ b, a, d } = { d, a, b }

but { b, a, x } ≠ { y, b, a }.

In the definition of a set, "distinct" means that we cannot have repeated objects – every object within a set is different from all the others in that set.

So, for example, you cannot have a set like { h, v, v, b, b, b }.

We take for granted the existence of the empty set denoted by ∅. This set contains no elements at all.

We can write ∅ = { }.

The “**union**” of two sets is the set of elements which belong to either one of the two sets – any repeated elements are simply dropped. We write:

$A \cup B$ for the union of the two sets A and B.

$A \cup B = \{ x: x \in A \textit{ or } x \in B \}$.

The word “*or*” in the above statement (and indeed throughout mathematics) is the “inclusive or” – it means “one or the other, or both[1]”.

Hence:

$\{ a, e, o \} \cup \{ o, u, a, i \} = \{ a, e, i, o, u \}$

$A \cup \emptyset = A$,

$A \cup A = A$,

$A \cup B = B \cup A$. See footnote[2] below.

The “**intersection**” of two sets, A and B is the set of elements which belong to *both* sets and is written $A \cap B$. Hence:

$A \cap B = \{ x: x \in A \textit{ and } x \in B \}$.

For example, if $A = \{ b, o, l, k, s \}$ and $B = \{ b, a, l, s \}$ then

$A \cap B = \{ b, l, s \}$.

As further examples, note that, for any set A:

$A \cap A = A$,

$A \cap \emptyset = \emptyset$.

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[1] Offer a mathematician a cake or a bun and he will feel free to take both (*she* will probably decline both and nibble on a lettuce leaf to preserve her alluring figure!)

[2] Mathematicians call this the “commutative” property. There is also the “associative property” which says that $(A \cup B) \cup C = A \cup (B \cup C)$ – in other words, if you are uniting three sets, it doesn’t matter which order you unite them in.

The last concept to mention is that of a "**subset**". The set B is said to be a subset of the set A if every element of B belongs to the set A and we write $B \subset A$ or, equivalently, $A \supset B$.

Note that, for any set A:

$A \subset A$ (every set is a subset of itself),

$\emptyset \subset A$ (the empty set is a subset of every set),

If $A \subset B$ and $B \subset A$, then $A = B$.

With just these few notions an amazingly rich theory can be constructed – which provides the foundation for the whole of mathematics (as shown by Bertand Russell and A.N. Whitehead in their book "Principia Mathematica" – but they had a few difficulties along the way . . .)

Before we leave this area, we should note that there is a set of axioms for set theory (actually more than one, but the Zermelo-Frankel axioms are those now commonly adopted). Most interestingly it is possible to construct the natural numbers from set theory, thereby rendering the Peano axioms redundant.

4. Notation for the Set of all Natural Numbers

$\mathbb{N}$ (a specially adapted form of the letter N) is usually used to denote the set of all natural numbers.

$$\mathbb{N} = \{1, 2, 3, . . . \}.$$

Now we can do **some** arithmetic with the natural numbers. We can add any two of them and get another one. We can multiply any two natural numbers and we shall get another natural number. Subtraction, however can be problematical! Certainly we can subtract 6 from 9 and get another natural number – but what about subtracting 9 from 6 or even 9 from 9? We need more numbers!

5. The Integers

If we throw 0 and the negative counting numbers into the bag we get a set known as the integers, always denoted by $\mathbb{Z}$ (from the German "Zahlen" meaning numbers).

$$\mathbb{Z} = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}$$

Now we can cheerfully add, subtract and multiply the elements of this set and always get another element of the set – the set is "closed" under the operations of addition, subtraction and multiplication. What about division however? Sometimes it works, but mostly it doesn't. -6/3 is OK but 7/2 is not – and neither is it any good trying to divide any number divided by zero!

Note that: $\mathbb{N} \subset \mathbb{Z}$.

6. The Rationals

Let's invent the set $\mathbb{Q}$ of "rational numbers":

$$\mathbb{Q} = \{ p/q : p, q \in \mathbb{Z}, q \neq 0\}.$$

In other words, $\mathbb{Q}$ is the set of all "fractions" - the set of all Quotients (or Ratios) of any two integers, provided that the denominator is not zero.

Examples are: 1/2, -356/23, 3/3, -6/-6

Note that the term "rational" merely refers to the fact that these are RATIOS of integers – they are not endowed with any special quality of reasonableness!

Note that we can solve any equation of the form n . y = m, where y is an unknown number and m, n $\in \mathbb{Z}$. For example, given 3 . y = 4 we can deduce that y = 4/3.

Note that $\mathbb{Z} \subset \mathbb{Q}$.

7. Decimal Representation of Rational Numbers

Note that all decimal numbers (for example 3.14) are simply rational numbers written with standard denominators – denominators which are powers of 10.

Thus, 3.14 = 314/100

A small difficulty arises when we are trying to express a number like 1/3 as a decimal – and we come out with 0.3333… (where the 3 recurs indefinitely). Another example is

1/7 = 0.142857142857142857. . . where the group "142857" repeats indefinitely.

I haven't time to deal further with this in detail, but will make mention of it in an aside on infinite series.

8. The Number Line

Consider a ruler – extended infinitely in both directions. It is marked with a zero. One inch to the right of zero we mark the number 1, two inches to the right of zero is marked the number 2 and so on. The negative numbers are analogously marked.

__________|_______|_______|_______|_______| _______|_______|_______|________

-3 -2 -1 0 1 2 3 4

Let's now start filling in the gaps with rational numbers. Concentrate on the interval between 0 and 1, for all other segments can be similarly filled. We can put in ½, 1/3, 2/3, ¼, ¾ and so on.

_______________0 ___1/4___ 1/3 ______1/2______________1________

It's pretty obvious that we can fill this interval (and, indeed, the whole line) as densely as we wish – and the question then arises: Does the set of rational numbers, $\mathbb{Q}$ (which is infinite, of course), actually fill the whole line?

Amazingly enough (to me, at least) the answer is "NO" – there is a hole in it!

Consider the number √2 (the number which, when multiplied by itself is precisely 2).

I will give you a million pounds if you can find a rational number whose square is 2!

Let's try a few things:Clearly the answer if it exists) must lie between 1 and 2 because $1^2 = 1$ and $2^2 = 4$. How about 1.5? Well, $1.5^2 = 2.25$ = so go lower – and so on.

Number	Number Squared
1	1
2	4
1.5	2.25
1.4	1.96
1.41	1.9881
1.4142	1.999961640
1.414213562	1.999999998944727844

9. Proof that √2 is not a rational number

The proof is by contradiction – that is, we assume that there is a rational number whose square is 2 and show that this leads to absurdity.

Suppose that a/b is a rational number whose square is 2. If there is a such a number, then there exists one IN LOWEST TERMS (i.e. which has been "reduced" so that there is no common factor in both the numerator and denominator. We do this by the familiar process of "cancellation").

So now, there is (we assume) a rational number IN LOWEST TERMS whose square is exactly 2.

Let this number be p/q, so that:

$$(p/q)^2 = 2 \qquad (1)$$

$$\text{Then} \quad \frac{p^2}{q^2} = 2 \qquad (2)$$

$$\text{So} \quad p^2 = 2 \times q^2 \qquad (3)$$

But this implies that p is an EVEN number (the right hand side of the above equation is even and if we square an ODD number we get another odd number).

Hence we can write p in the form 2 x r for some integer r. Substituting this in equation (3) gives:

$$4 \times r^2 = 2 \times q^2 \qquad (4)$$

Dividing both sides of equation (4) by 2 gives:

$$2 \times r^2 = q^2 \qquad (5)$$

But this means that q must be an even number.

Hence now BOTH p and q are even and this contradicts the fact that p and q have no common factor.

Hence there is no rational number whose square is 2!

Let's look at Mathematica - an amazing program for doing mathematics.

Try N[Sqrt[2],20] and N[Sqrt[2],5000].

10. Irrational Numbers

We have (to your horror, no doubt) discovered that there is a tiny hole in the number line! This hole lies between 1.414 and 1.415 (though we can refine this to any degree of accuracy we wish).

Although $\sqrt{2}$ might seem like an oddity, there are many other examples – what about $\sqrt{3}$ and $\sqrt{5}$ for example?

It turns out that there are infinitely many holes in the number line. To make it whole again mathematicians defined the concept of "irrational numbers". They are not lacking in reasonableness, nor prone to quirky behaviour - they are simply "not rational". All numbers on the number line that are not rational are thus called "irrational".

What is even more bizarre is that (in a sense to be defined) there are more holes than dots (assuming we have dotted in the rational numbers) in the number line! LOTS MORE!

11. Real Numbers

The totality (or "continuum") of numbers on the number line are called "The Real Numbers" and the set of them is denoted by $\mathbb{R}$. Now we can solve equations such as $x^2 = 2$.

We now have: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

12. Complex Numbers

Before leaving the field of numbers it should be noted that we cannot find real-number solutions of all "algebraic equations" – ones like $5x^3 + 3x^2 – 7x + 3 = 0$. We cannot even find a real number solution to $x^2 + 1 = 0$. The solution is to continue inventing sets of numbers – this time we invent the "complex numbers" – these are of the form $x + iy$ where x and y are real numbers and I is the square root of minus one.

This is a huge and fascinating subject. Complex numbers provide a means of solution of many practical problems – including many in electrical engineering and in aerodynamics.

It is tempting to think that we can continue to invent "larger" (or at least more comprehensive) sets of numbers – but it turns out that this is unnecessary – at least in arithmetic. The complex numbers allow us to describe (and find!) solutions to all algebraic equations.

13. Comparing the Sizes of Sets

I have two jars with a number of distinct objects in each. How can I determine which jar has the greater number of objects in it (or if they have the same number of objects in both).

Method I:

The usual answer is to count the number in each jar.

Method 2:

An alternative method is to match the objects one for one (if you like, remove one object from each jar – pair them up, perhaps even joining them with a short piece of string – and see which jar has objects left over (if no objects left over, then clearly there were an equal number of objects in each jar.

As another example: Consider a dance. Are there more females than males? Let everyone who can pair off with a member of the opposite sex (not very politically correct, I suppose). Are there any wallflowers?

Let A = {Button, Paper Clip, Coin, Rubber Band}

and

B = {Safety Pin, Cotton Reel, Pebble, Sweetie, Die}

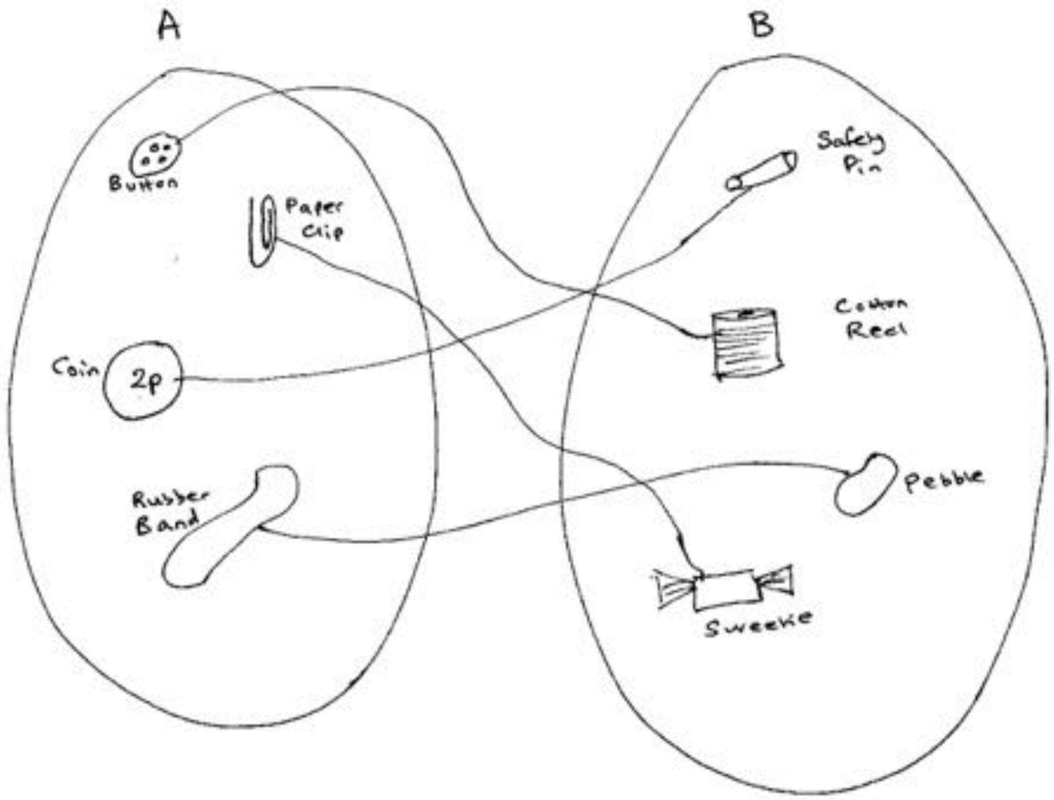


Diagram 1.

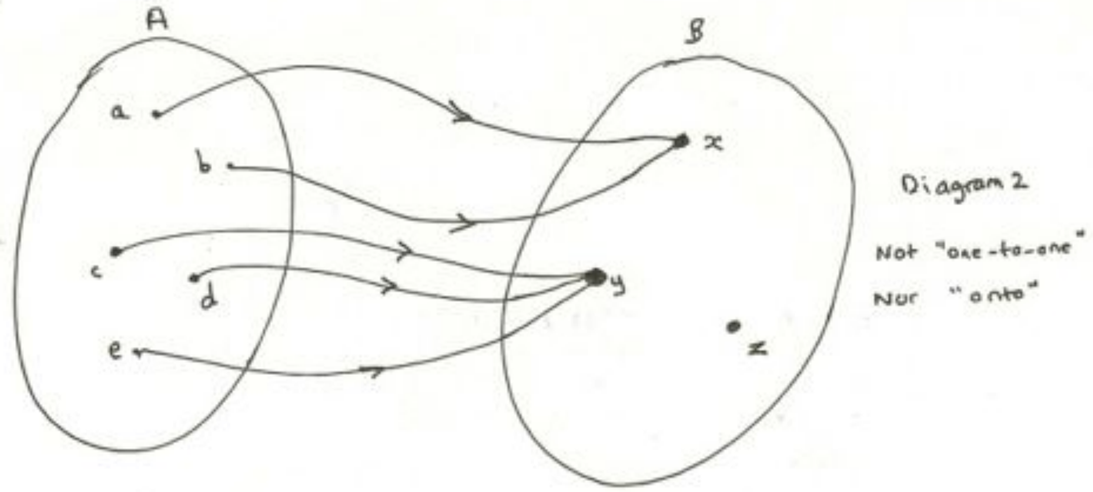
Both methods clearly work for counting finite sets, but method 1 fails for infinite sets. One of Georg Cantor's great insights was to use method 2 for comparing the sizes of infinite sets. Some of the results are startlingly counter-intuitive – so much so that Cantor was considered insane by most mathematicians of his day! (Nowadays this "insanity" is embraced by all mathematicians).

In more technical terms we say that if there is a "**bijection**" between two sets, then the sets have the same number of elements. A bijection is simply a mapping that is **one-to-one** and "**onto**". A "mapping" (or "function") associates with every element of one set a definite element of another set.

Diagrams here illustrating functions:

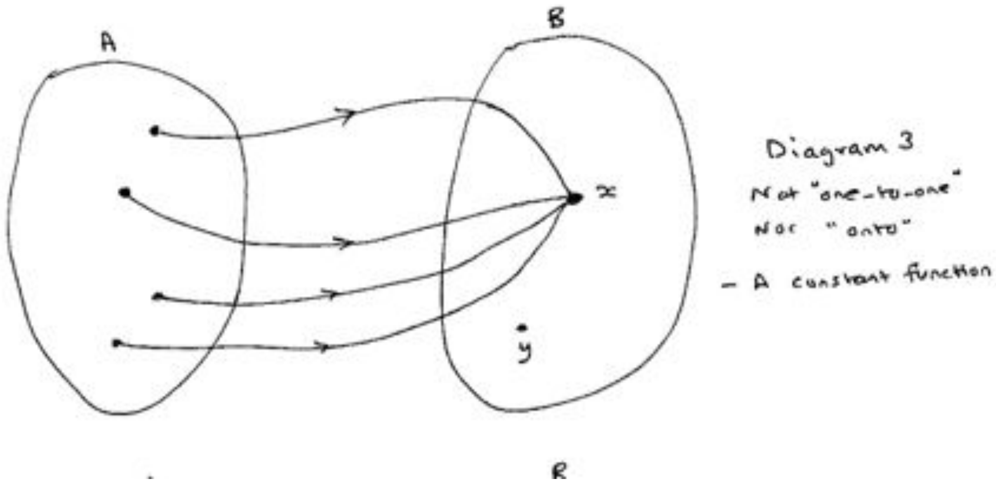
Function from set A → set B which is not "one-to-one" nor "onto" (Diagram 2)

f(a) = x, f(b) = x, f(c) = y,   f(d) = y, , f(e) = y.

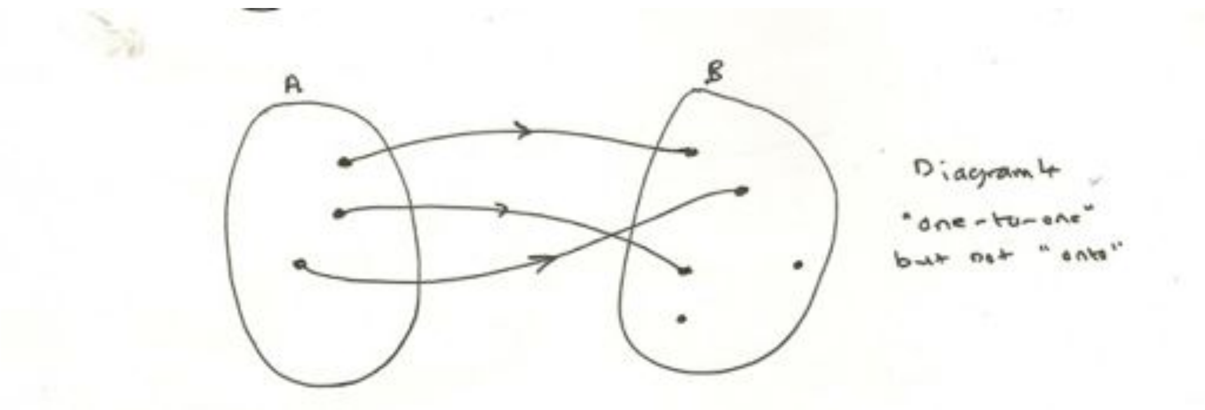


Function from set A → set B which is not "one-to-one" nor "onto" (Diagram 3)

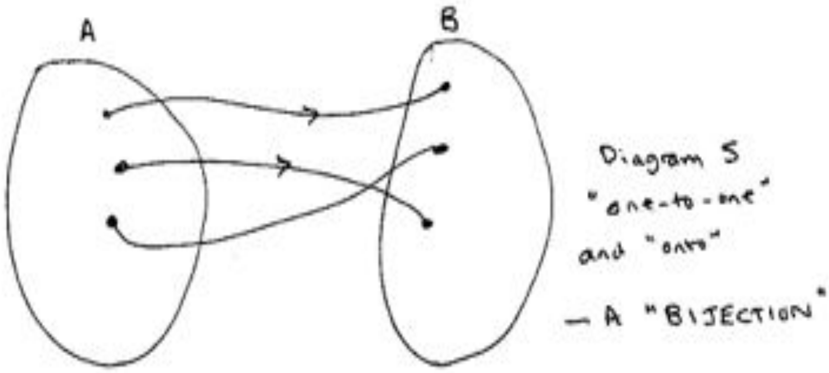
f(r) = x for all r belonging to A (a "constant" function)



Function from set A → set B which **is** "one-to-one" but not "onto" (Diagram 4)



Function from set A → set B which **is "**one-to-one" and "onto" (Diagram 5)..



14. There are the Same Number of Even Integers as there are Integers

Let E be the set of all even integers, so E = { . . . , -6, -4, -2, 0, 2, 4, 6, . . . }

Consider the function f: $\mathbb{Z} \rightarrow$ E defined by f(a) = 2a. This is clearly a bijection.

15. There are the Same Number of Rational Numbers as there are Integers?

We will restrict ourselves to comparing the positive rational numbers, $\mathbb{Q}^+$ with the Natural Numbers $\mathbb{N}$ = { 1, 2, 3, . . .}. The argument can be fairly easily extended to comparing $\mathbb{Q}$ with $\mathbb{Z}$.

We need a clever way of listing "all the fractions" and then "counting them":

p \ q	1	2	3	4	5	.	.	.
1	1/1	1/2	1/3	1/4	1/5	.	.	.
2	2/1	2/2	2/3	2/4	2/5	.	.	.
3	3/1	3/2	3/3	3/4	3/5	.	.	.
4	4/1	4/2	4/3	4/4	4/5	.	.	.
5	5/1	5/2	5/3	5/4	5/5	.	.	.
.								
.								
.								

This infinite table contains every element of $\mathbb{Q}$.

We count them in a "square fashion". This is equivalent to a bijection from $\mathbb{Q} \rightarrow \mathbb{N}$ defined by:

f(p/q) = $p^2 - q + 1$ whenever p is greater than or equal to q

and

$(q - 1)^2 + p$ whenever p is less than q.

So $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are all the same size! We call the size of a set its “Cardinality” and denote the cardinal number of these infinite sets as $\aleph_0$ (aleph null or aleph nought. The Aleph is the first letter of the Hebrew alphabet).

Aside – consider replacing the “10” in “There were 10 green bottles hanging on a wall” with $\aleph_0$.

16. Are All Infinite Sets the Same Size?

By now you might be tempted to think that all infinite sets are the same size – that they are all “countable”. Amazingly enough it turns out that this is not so. It can be proved that there are “far more” real numbers that there are natural numbers (or integers or rationals).

Cantor proved that it is not possible to construct a bijection between $\mathbb{R}$ and $\mathbb{N}$. I will outline the idea behind Cantor’s “diagonal argument” on the flip chart – there are some technicalities which are a little beyond the scope of this lecture - but if you have understood everything so far you should be able to follow the argument (which you can find in Wikipedia).

The Cardinality (size) of $\mathbb{R}$ is denoted by $\mathfrak{c}$ - a letter “c” (for “continuum”) in lower case German font.

17. Are There Even Larger Sets than the Real Numbers?

It turns out that "larger" sets can always be constructed (ones for which no bijection is possible with any "smaller" set). The key to this lies in examining subsets.

How many subsets does a set have? For finite sets the answer is easy – if a set has n elements there are always 2^n subsets. For example the set S = {a, b, c} has the following eight subsets:

{ }	(the empty set),
{a}, {b}, {c},	(three sets with one element in each)
{a, b}, {a, c}, {b, c},	(three sets with two elements in each)
{a, b, c}	(the entire set, S, which from our definition is a subset of itself).

The set of all subsets of a given set is called its "Power Set". It is clear that the power set of a finite set is always larger than the set itself.

It is possible(though perhaps a little mind-boggling) to consider the set of all subsets of an infinite set and to show that this "power set" is always of larger cardinality than the set itself. Hence there are always bigger and bigger infinities!

Try to get your head around the set of all subsets of the natural numbers! This power set has the same size as the Real Numbers.

18. Is there a Set which is Intermediate in Size Between $\aleph_0$ and $\mathfrak{c}$?

That there is no such set is the famous "Continuum Hypothesis" which is one of the great unsolved problems of mathematics. Rather like the parallel postulate in Euclidean Geometry, it has been shown that the Continuum Hypothesis cannot be proved or disproved using the (Zermelo-Fraenkel) axioms of set theory. Actually, there is a proviso – that those axioms are consistent (not self-contradictory) and this is itself a bit of an open question . . .

19. Transcendental Numbers

Just a passing curiosity – Pi is not an algebraic number. Neither is e. These irrationals are called "transcendental". There are lots of them . . . in fact there are as many as there are Real Numbers!

Like all irrationals they are infinite non-repeating decimals.

Mathematica – digits of Pi.

20. Infinite Series

Consider:

S_1 = 1 + 1/2 + 1/4 + 1/8 + . . . (Convergent)

S_2 = 1 + 1/2 + 1/3 + 1/4 + 1/5 + . . . (the Harmonic Series)

Where the sums (if they exist) are meant to take as many terms as we wish.

The first series is a geometric series (as you will recall from schooldays!) with "common ratio" = ½ (that is, each term is obtained by multiplying its predecessor by ½).

Trial and error will lead you to two conclusions:

(a) The sum of the series never exceeds 2, no matter how many terms we add in
(b) We can make the sum as close to 2 as we wish by adding more and more terms.

We say that the first series is "convergent" and "converges to a limit of 2". There is a way to make this limit concept completely precise and rigorous, but that is beyond our scope. Just note that if we are not too rigorous we can do some magic with algebra to come out with the formula for the sum of a geometric series.

The second series (which is known as the Harmonic Series) is much more mysterious. Like the first, its terms get smaller and smaller. Nevertheless, this sum of this series just goes on getting bigger the more terms we add it. It is "divergent" (and this is not hard to prove). It diverges incredibly slowly – for example, to exceed a sum of 5 we need 83 terms. After ten million terms we have only reached 16.695. It is a VERY slowly divergent series.

The series has a physical analogue with a pack of cards. I will demonstrate.